

# Stochastic Well-posed Systems and Well-posedness of Some Stochastic Partial Differential Equations with Boundary Control and Observation

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## Abstract

We generalize the concept “well-posed linear system” to stochastic linear control systems and study some basic properties of such kind systems. Under our generalized definition, we show the well-posedness of the stochastic heat equation and the stochastic Schrödinger equation with suitable boundary control and observation operators, respectively.

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**Key Words.** stochastic well-posed linear system, stochastic heat equation, stochastic Schrödinger equation.

## 1 Introduction

Let  $H$ ,  $U$  and  $\tilde{U}$  be three Hilbert spaces which are identified with their dual spaces. Let  $A$  be the generator of a  $C_0$ -semigroup  $\{S(t)\}_{t \geq 0}$  on  $H$ . Denote by  $H_{-1}$  the completion of  $H$  with respect to the norm  $|x|_{H_{-1}} \triangleq |(\beta I - A)^{-1}x|_H$ , where  $\beta \in \rho(A)$  is fixed. Let  $B \in \mathcal{L}(U, H_{-1})$  and  $\mathcal{C} \in \mathcal{L}(D(A), \tilde{U})$ .

Consider the following control system:

$$\begin{cases} \frac{dY(t)}{dt} = AY(t) + Bu(t) & \text{in } (0, +\infty), \\ Y(0) = Y_0, \\ Z(t) = CY(t) & \text{in } (0, +\infty). \end{cases} \quad (1.1)$$

Here  $Y_0 \in H$  and  $u \in L^2(0, +\infty; U)$ .

In (1.1), the expected state space is  $H$ . If  $B \in \mathcal{L}(U, H)$ , one can easily show that  $Y(t) \in H$  for all  $t \geq 0$  by the classical theory of evolution equations (see, for instance, [2, Chapter 3]).

If  $\mathcal{C} \in \mathcal{L}(H, \tilde{U})$ , then the observation  $CY(t)$  makes sense for all  $H$ -valued function  $Y(\cdot)$ .

However, in practical control system, it is very common that the control and observation operators are unbounded with respect to the state space. Typical examples are systems governed by partial differential equations (PDEs) in which actuators and sensors act on lower-dimensional hypersurfaces or on the boundary of a spatial domain.

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The unboundedness of the control and observation operators leads to substantial technical difficulties even for the formulation of the state space. For instance, when  $B \in \mathcal{L}(U, H_{-1})$ , it seems that the natural state space for the dynamics of (1.1) should be  $H_{-1}$ . On the other hand, to handle the unbounded observation operator  $\mathcal{C}$ , the natural state space should be  $D(A)$ .

In the deterministic setting, to overcome the above gap between the state spaces, people introduced the notions of the admissible control operator and the admissible observation operator (see [21, 22] for example). Furthermore, to study the feedback control problem, people introduced the notion of “well-posed linear system”, which satisfies, roughly speaking, that the map from the input space  $L^2(0, T; U)$  to the output space  $L^2(0, T; \tilde{U})$  is bounded ([22, 25, 26, 27]).

The well-posed linear systems form a very general class whose basic properties are rich enough to develop a parallel theory for the theory of control systems with bounded control and observation operators, such as feedback control, dynamic stabilization and tracking/disturbance rejection. Furthermore, the well-posed linear systems are quite general and covers many control systems described by partial differential equations with actuators and sensors acting on lower-dimensional hyper-surfaces or on the boundary of a spatial domain, which is the starting point of the study of such kind of systems.

As far as we know, there are a lot of references concerning the well-posed deterministic linear systems, both for abstract control system and controlled PDEs in the last three decades (see [1, 4, 5, 6, 7, 25, 24, 29, 30] and the rich references therein). However, to our best knowledge, there is a blank field for the well-posedness of the stochastic linear systems. In this paper, we generalize the notion of well-posed linear system to the stochastic context by providing a formulation of the stochastic well-posed linear system and some basic properties.

We first introduce some notations. Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a complete filtered probability space with a filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ , on which a one-dimensional standard Brownian motion  $\{W(t)\}_{t \geq 0}$  is defined. Let  $\mathcal{X}$  be a Banach space. For any  $p \in [1, \infty)$ ,  $t \geq 0$  and a sub- $\sigma$ -algebra  $\mathcal{M}$  of  $\mathcal{F}$ , denote by  $L_{\mathcal{M}}^p(\Omega; \mathcal{X})$  (resp.  $L_{\mathcal{M}}^\infty(\Omega; \mathcal{X})$ ) the set of all  $\mathcal{M}$ -measurable ( $\mathcal{X}$ -valued) random variables  $\xi : \Omega \rightarrow \mathcal{X}$  with  $\mathbb{E}|\xi|_{\mathcal{X}}^p < \infty$  (resp.  $\text{esssup}_{\omega \in \Omega} |\xi|_{\mathcal{X}} < \infty$ ). Next, for any  $p, q \in [1, \infty)$  and  $T \in (0, +\infty]$ , put

$$\begin{aligned} L_{\mathbb{F}}^p(0, T; L^q(\Omega; \mathcal{X})) &\triangleq \left\{ \varphi : (0, T) \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \int_0^T \left( \mathbb{E}|\varphi(t)|_{\mathcal{X}}^q \right)^{\frac{p}{q}} dt < \infty \right\}, \\ L_{\mathbb{F}}^\infty(0, T; L^q(\Omega; \mathcal{X})) &\triangleq \left\{ \varphi : (0, T) \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \text{esssup}_{t \in [0, T]} \mathbb{E}|\varphi(t)|_{\mathcal{X}}^q < \infty \right\}, \\ L_{\mathbb{F}}^p(0, T; L^\infty(\Omega; \mathcal{X})) &\triangleq \left\{ \varphi : (0, T) \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \int_0^T \left[ \text{esssup}_{\omega \in \Omega} |\varphi(t)|_{\mathcal{X}} \right]^p dt < \infty \right\}, \\ L_{\mathbb{F}}^\infty(0, T; \mathcal{X}) &\triangleq \left\{ \varphi : (0, T) \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \text{esssup}_{(t, \omega) \in [0, T] \times \Omega} |\varphi(t)|_{\mathcal{X}} < \infty \right\}, \\ L_{Br}^p(0, T; \mathcal{X}) &\triangleq \left\{ \varphi : (0, T) \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is Borel measurable and } \int_0^T \left( |\varphi(t)|_{\mathcal{X}}^p \right) dt < \infty \right\}. \end{aligned}$$

When  $p = q$ , we simply denote  $L_{\mathbb{F}}^p(0, T; L^p(\Omega; \mathcal{X}))$  by  $L_{\mathbb{F}}^p(0, T; \mathcal{X})$ . Further, for any  $p \in [1, \infty)$ , set

$$C_{\mathbb{F}}([0, T]; L^p(\Omega; \mathcal{X})) \triangleq \left\{ \varphi : [0, T] \times \Omega \rightarrow \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \mathbb{E}(|\varphi(t)|_{\mathcal{X}}^p)^{\frac{1}{p}} \text{ is continuous} \right\}.$$

All the above spaces are endowed with the canonical norms.

Consider the following stochastic control system:

$$\begin{cases} dY(t) = [A + F_1(t)]Y(t)dt + Bu(t)dt + F_2(t)Y(t)dW(t) & \text{in } (0, +\infty), \\ Y(0) = Y_0, \\ Z(t) = CY(t) & \text{in } (0, +\infty). \end{cases} \quad (1.2)$$

Here  $Y_0 \in L^2_{\mathcal{F}_0}(\Omega; H)$ ,  $u \in L^2_{\mathbb{F}}(0, T; U)$  and  $F_1, F_2 \in L^\infty_{\mathbb{F}}(0, +\infty; \mathcal{L}(H))$ .

We first give the definition of the mild solution to (1.2).

**Definition 1.1** *An  $H$ -valued stochastic process  $Y(\cdot)$  is called a mild solution to (1.2) if*

1.  $Y(\cdot) \in C_{\mathbb{F}}([0, +\infty); L^2(\Omega; H))$ ;
2. For any  $t \in [0, +\infty)$ ,

$$\begin{aligned} Y(t) = & S(t)Y_0 + \int_0^t S(t-s)F_1(s)Y(s)ds + \int_0^t S(t-s)Bu(s)ds \\ & + \int_0^t S(t-s)F_2(s)Y(s)dW(s). \end{aligned}$$

In general, the stochastic convolution  $\int_0^t S(t-s)F_2(s)Y(s)dW(s)$  is no longer a martingale. Then, we cannot apply Itô's formula to mild solutions directly on most occasions. This will limit the way to establish suitable energy estimate for stochastic partial differential equations. To avoid this restriction, the notion of weak solution is introduced.

**Definition 1.2** *A process  $Y(\cdot) \in C_{\mathbb{F}}([0, +\infty); L^2(\Omega; H))$  is called a weak solution to (1.2) if for all  $t \in [0, +\infty)$  and  $\psi \in D(A^*)$ ,*

$$\begin{aligned} \langle Y(t), \psi \rangle_H = & \langle Y_0, \psi \rangle_H + \int_0^t \langle Y(s), A^*\psi \rangle_H ds + \int_0^t \langle F_1(s)Y(s), \psi \rangle_H ds + \int_0^t \langle Y(s), B^*\psi \rangle_H ds \\ & + \int_0^t \langle F_2(s)Y(s), \psi \rangle_H dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

**Remark 1.1** *If  $B \in \mathcal{L}(U, H)$ , then it is well-known that a mild solution to (1.2) is also a weak solution to (1.2) (see [3] for example). However, when  $B \in \mathcal{L}(U, H_{-1})$ , as far as we know, there is no result for the relationship between these two kinds of solutions.*

The rest of the paper is organized as follows: in Section 2, we give the formulation of stochastic well-posed linear system and some basic properties about it. Sections 3 and 4 are devoted to the study the well-posedness of controlled stochastic heat equations and Schrödinger equations, respectively. In Section 5, we give some further comments and present some open problems.

Please note that in order to present the key idea in a simple way, we do not pursue the full technical generality in this paper.

## 2 Formulation and basic properties of stochastic well-posed linear systems

In this section, we give the formulation of a stochastic well-posed linear system and study some of its basic properties. First, we recall the notion of the admissible control operator, which is first introduced in the context of deterministic control systems. Then we show the existence and uniqueness of the mild and weak solutions to (1.1) when  $B$  is an admissible control operator. Next, we recall the concept of the admissible observation operator. At last, we present the definition of the stochastic well-posed linear system.

## 2.1 Admissible control operator

The concept of admissible control operator is motivated by the study of the solution to the deterministic control system (1.1), where, people would like to study the operator  $B$  for which all the mild solutions  $Y$  to (1.1) belong to  $C([0, +\infty); H)$ . Such operator is called admissible. In this paper, we will show that it is also a suitable notion when studying the solution to stochastic control system (1.2).

Let  $t \in [0, +\infty)$ . We define an operator  $\Phi_t \in \mathcal{L}(L^2_{\mathbb{F}}(0, +\infty; U), L^2_{\mathcal{F}_t}(\Omega; H_{-1}))$  by

$$\Phi_t u = \int_0^t S(t-s) B [\chi_{[0,t]}(s) u(s)] ds. \quad (2.1)$$

**Definition 2.1** *The operator  $B \in \mathcal{L}(U, H_{-1})$  is called an admissible control operator (for  $\{S(t)\}_{t \geq 0}$ ) if there is a  $t_0 > 0$  such that  $\text{Range}(\Phi_{t_0}) \subset L^2_{\mathcal{F}_{t_0}}(\Omega; H)$ .*

**Remark 2.1** *As far as we know, the concept of admissible control operator was first introduced in [8] in the deterministic framework. Soon afterwards, it was presented systematically as an ingredient of the well-posed linear system in [22]. Our definition of admissible control operator for stochastic control systems enjoys the same spirit (see Proposition 2.1 for the detail). However, we formulate it as Definition 2.1 for the convenience of the study of stochastic control problems.*

**Remark 2.2** *Clearly, if  $B$  is admissible, then in (2.1), the integrand takes values in  $L^2_{\mathcal{F}}(\Omega; H_{-1})$ , but the resultant integral lies in  $L^2_{\mathcal{F}}(\Omega; H)$ , a dense subspace of  $L^2_{\mathcal{F}}(\Omega; H_{-1})$ . As for the deterministic case, if  $B \in \mathcal{L}(U, H)$ , then  $B$  is admissible.*

**Remark 2.3** *People had proven the admissibility for some control operators originated in controllability problems of stochastic PDEs with boundary control (see [15, 17] for example). In this paper, we will prove that the control operators in controlled stochastic heat equations and Schrödinger equations with suitable boundary control are admissible.*

We have the following properties for the admissible control operator.

**Proposition 2.1** *The control operator  $B \in \mathcal{L}(U, H_{-1})$  is admissible if and only if there is a constant  $C = C(t_0) > 0$  such that for any  $u \in L^2_{\mathbb{F}}(0, +\infty; U)$ ,*

$$|\Phi_{t_0} u|_H \leq C \int_0^{t_0} |u(s)|_U^2 ds, \quad \mathbb{P}\text{-a.s.} \quad (2.2)$$

*Proof:* The “if” part is obvious. Let us prove the “only if” part. We do it by contradiction argument. If (2.2) were untrue, then there is a sequence  $\{u_n\}_{n=1}^\infty \subset L^2_{Br}(0, +\infty; U) \subset L^2_{\mathbb{F}}(0, +\infty; U)$  with  $|u_n|_{L^2_{Br}(0, +\infty; U)} = 1$  such that

$$|\Phi_{t_0} u_n|_H \geq n,$$

that is,

$$|\Phi_{t_0} u_n|_{L^2_{\mathcal{F}_{t_0}}(\Omega; H)} \geq n. \quad (2.3)$$

Let  $\lambda \in \rho(A)$  and  $B_0 = (\lambda I - A)^{-1} B$ . Since  $B \in \mathcal{L}(U, H_{-1})$ , we have that  $B_0 \in \mathcal{L}(U, H)$  and

$$\Phi_{t_0} u = (\lambda I - A) \int_0^{t_0} S(t_0 - s) B_0 [\chi_{[0, t_0]}(s) u(s)] ds.$$

This implies that  $\Phi_{t_0}$  is closed. According to the closed-graph theorem, we know that  $\Phi_{t_0}$  is bounded, i.e., there is a constant  $C(t_0)$  such that for any  $u_n \in L_{B^r}^2(0, +\infty; U)$ ,

$$|\Phi_{t_0} u_n|_{L_{\mathcal{F}_{t_0}}^2(\Omega; H)} \leq C(t_0) |u_n|_{L_{\mathbb{F}}^2(0, T; U)},$$

a contradiction.  $\square$

**Proposition 2.2** *If  $\text{Range}(\Phi_{t_0}) \subset L_{\mathcal{F}_{t_0}}^2(\Omega; H)$  for a specific  $t_0 \geq 0$ , then for every  $t > 0$ ,  $\Phi_t \in \mathcal{L}(L_{\mathbb{F}}^2(0, +\infty; U), L_{\mathcal{F}_t}^2(\Omega; H))$ .*

*Proof:* Let  $u \in L_{\mathbb{F}}^2(0, +\infty; U)$ . It is easy to see that

$$\begin{aligned} \Phi_{2t_0} u &= \int_0^{2t_0} S(2t_0 - s) B[\chi_{[0, 2t_0]}(s) u(s)] ds \\ &= S(t_0) \int_0^{t_0} S(t_0 - s) B[\chi_{[0, t_0]}(s) u(s)] ds + \int_{t_0}^{2t_0} S(2t_0 - s) B[\chi_{[t_0, 2t_0]}(s) u(s)] ds \\ &= S(t_0) \Phi_{t_0} u + \Phi_{t_0} \tilde{u}. \end{aligned}$$

where  $\tilde{u}(s) = u(t_0 + s)$ . According to Proposition 2.1, we find that

$$\mathbb{E}|\Phi_{2t_0} u|_H^2 = \mathbb{E}|S(t_0) \Phi_{t_0} u + \Phi_{t_0} \tilde{u}|_H^2 \leq C(|u|_{L_{\mathbb{F}}^2(0, t_0; U)}^2 + \mathbb{E}|\tilde{u}|_{L^2(0, t_0; U)}^2) \leq C|u|_{L_{\mathbb{F}}^2(0, 2t_0; U)}^2.$$

This deduces that  $\Phi_{2t_0} \subset \mathcal{L}(L_{\mathbb{F}}^2(0, +\infty; U), L_{\mathcal{F}_{2t_0}}^2(\Omega; H))$ . By induction, for all  $n \in \mathbb{N}$ ,  $\Phi_{2^n t_0} \subset \mathcal{L}(L_{\mathbb{F}}^2(0, +\infty; U), L_{\mathcal{F}_{2^n t_0}}^2(\Omega; H))$ .

For any  $t > 0$ , there is a  $n \in \mathbb{N}$  such that  $t \in (0, 2^n t_0]$ . For  $u \in L_{\mathbb{F}}^2(0, +\infty; U)$ , let

$$\tilde{u}(s) = \begin{cases} 0, & \text{if } s \in [0, 2^n t_0 - t), \\ u(s - t), & \text{if } s \in [2^n t_0 - t, +\infty). \end{cases}$$

Then, we have that

$$\begin{aligned} \Phi_t u &= \int_0^t S(t - s) B[\chi_{[0, t]}(s) u(s)] ds \\ &= \int_{2^n t_0 - t}^{2^n t_0} S(2^n t_0 - s) B[\chi_{[2^n t_0 - t, 2^n t_0]}(s) \tilde{u}(s)] ds \\ &= \Phi_{2^n t_0} \tilde{u}. \end{aligned}$$

Hence, we get that

$$\mathbb{E}|\Phi_t u|_H^2 = \mathbb{E}|\Phi_{2^n t_0} \tilde{u}|_H^2 \leq C|\tilde{u}|_{L_{\mathbb{F}}^2(0, 2^n t_0; U)}^2 = C|u|_{L_{\mathbb{F}}^2(0, t; U)}^2,$$

which concludes that  $\Phi_t \in \mathcal{L}(L_{\mathbb{F}}^2(0, +\infty; U), L_{\mathcal{F}_t}^2(\Omega; H))$ .  $\square$

**Proposition 2.3** *If  $B$  is an admissible control operator, then the mapping*

$$\begin{cases} \Lambda : (0, +\infty) \times L_{\mathbb{F}}^2(0, +\infty; U) \rightarrow L_{\mathcal{F}}^2(\Omega; H), \\ \Lambda(t, u) = \Phi_t u, \end{cases}$$

*is continuous.*

*Proof:* From Proposition 2.2, we know that for any  $t \in (0, +\infty)$ ,  $\Lambda(t, \cdot)$  is a continuous map from  $L^2_{\mathbb{F}}(0, +\infty; U)$  to  $L^2_{\mathcal{F}_t}(\Omega; H) \subset L^2_{\mathcal{F}}(\Omega; H)$ .

Next, we prove the continuity of  $\Lambda(t, u)$  with respect to  $t$ . Let  $u \in L^2_{\mathbb{F}}(0, +\infty; U)$  be fixed and put  $f(t) = \Phi_t u$ . Let  $0 < t_1 < t_2$ . By Lebesgue's dominated convergence theorem, we find that

$$\begin{aligned} & \lim_{t_2 \rightarrow t_1^+} \mathbb{E} |f(t_2) - f(t_1)|_H^2 \\ & \leq 2 \lim_{t_2 \rightarrow t_1^+} \mathbb{E} \left| \int_{t_1}^{t_2} S(t_2 - s) [\chi_{[0, t_2]}(s) u(s)] ds \right|_H^2 \\ & \quad + 2 \lim_{t_2 \rightarrow t_1^+} \mathbb{E} \left| \int_0^{t_1} [S(t_2 - t_1) - I] S(t_1 - s) [\chi_{[0, t_1]}(s) u(s)] ds \right|_H^2 \\ & = 0. \end{aligned}$$

This shows that  $f(\cdot)$  is right continuous. Similarly, we can show that  $f(\cdot)$  is left continuous. Hence,  $f(\cdot)$  is continuous.

The joint continuity of  $\Lambda(\cdot, \cdot)$  follows easily from the fact that

$$\Phi_t u - \Phi_s v = \Phi_t(u - v) + (\Phi_t - \Phi_s)v.$$

□

## 2.2 The existence and uniqueness of the mild and weak solution to (1.2)

From Proposition 2.3, we can prove the following result.

**Theorem 2.1** *Let  $B$  be an admissible control operator. Then the equation (1.2) admits a unique mild solution  $Y(\cdot) \in C_{\mathbb{F}}([0, +\infty); L^2(\Omega; H))$ . Moreover, for any  $T > 0$ , there is a constant  $C(T) > 0$  such that*

$$|Y|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; H))} \leq C(T) (|Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + |u|_{L^2_{\mathbb{F}}(0, +\infty; U)}). \quad (2.4)$$

Once we assume that  $B$  is admissible, the proof of Theorem 2.1 is very similar to the case that  $B \in \mathcal{L}(U, H)$ , which can be found in [3, Chapter 6]. We give it here not only for completeness but also for presenting how we utilize the fact that  $B$  is admissible.

*Proof of Theorem 2.1:* Let  $f, g \in L^2_{\mathbb{F}}(0, +\infty; H)$ . We claim that

$$\begin{aligned} Y(\cdot) & \triangleq S(\cdot)Y_0 + \int_0^\cdot S(t-s)f(s)ds + \int_0^\cdot S(t-s)Bu(s)ds \\ & \quad + \int_0^\cdot S(t-s)g(s)dW(s) \in C_{\mathbb{F}}([0, +\infty); L^2(\Omega; H)). \end{aligned} \quad (2.5)$$

First, we have that

$$\begin{aligned} & |Y(t)|_{L^2_{\mathcal{F}_t}(\Omega; H)} \\ & = \left| S(t)Y_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)g(s)dW(s) \right|_{L^2_{\mathcal{F}_t}(\Omega; H)} \\ & \leq |S(\cdot)Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + \left| \int_0^t S(t-s)f(s)ds \right|_{L^2_{\mathcal{F}_t}(\Omega; H)} + \left| \int_0^t S(t-s)Bu(s)ds \right|_{L^2_{\mathcal{F}_t}(\Omega; H)} \\ & \quad + \left| \int_0^t S(t-s)g(s)dW(s) \right|_{L^2_{\mathcal{F}_t}(\Omega; H)} \\ & \leq C(t) (|Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + |u|_{L^2_{\mathbb{F}}(0, t; U)} + |f|_{L^2_{\mathbb{F}}(0, t; H)} + |g|_{L^2_{\mathbb{F}}(0, t; H)}). \end{aligned} \quad (2.6)$$

Therefore, we find that

$$|Y(\cdot)|_{L_{\mathbb{F}}^{\infty}(0,T;L^2(\Omega;H))} \leq C(T)(|Y_0|_{L_{\mathcal{F}_0}^2(\Omega;H)} + |u|_{L_{\mathbb{F}}^2(0,T;U)} + |f|_{L_{\mathbb{F}}^2(0,T;H)} + |g|_{L_{\mathbb{F}}^2(0,T;H)}). \quad (2.7)$$

Further, for  $0 \leq t_1 \leq t_2 < \infty$ ,

$$\begin{aligned} & |Y(t_2) - Y(t_1)|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ &= \left| [S(t_2) - S(t_1)]Y_0 + \int_{t_1}^{t_2} S(t_2 - s)f(s)ds + \int_{t_1}^{t_2} S(t_2 - s)Bu(s)ds + \int_{t_1}^{t_2} S(t_2 - s)g(s)dW(s) \right. \\ & \quad + \int_0^{t_1} [S(t_2 - t_1) - I]S(t_1 - s)f(s)ds + \int_0^{t_1} [S(t_2 - t_1) - I]S(t_1 - s)Bu(s)ds \\ & \quad \left. + \int_0^{t_1} [S(t_2 - t_1) - I]S(t_1 - s)g(s)dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ &\leq \left| [S(t_2 - t_1) - I]S(t_1)Y_0 \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} + \left| \int_{t_1}^{t_2} S(t_2 - s)f(s)ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ & \quad + \left| \int_{t_1}^{t_2} S(t_2 - s)Bu(s)ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} + \left| \int_{t_1}^{t_2} S(t_2 - s)g(s)dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ & \quad + \left| \int_0^{t_1} S(t_1 - s)[S(t_2 - t_1) - I]f(s)ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} + \left| \int_0^{t_1} S(t_1 - s)[S(t_2 - t_1) - I]Bu(s)ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ & \quad + \left| \int_0^{t_1} S(t_1 - s)[S(t_2 - t_1) - I]g(s)dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)}. \end{aligned} \quad (2.8)$$

Since

$$\left| [S(t_2 - t_1) - I]S(t_1)Y_0 \right|_H \leq C|Y_0|_H,$$

by Lebesgue's dominated convergence theorem, we get that

$$\lim_{t_1 \rightarrow t_2^-} \left| [S(t_2 - t_1) - I]S(t_1)Y_0 \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} = 0. \quad (2.9)$$

Furthermore,

$$\begin{aligned} & \lim_{t_1 \rightarrow t_2^-} \left| \int_{t_1}^{t_2} S(t_2 - s)f(s)ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ &\leq \lim_{t_1 \rightarrow t_2^-} \int_{t_1}^{t_2} |S(t_2 - s)f(s)|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} ds \\ &\leq C \lim_{t_1 \rightarrow t_2^-} \int_{t_1}^{t_2} |f(s)|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} ds = 0 \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} & \lim_{t_1 \rightarrow t_2^-} \left| \int_{t_1}^{t_2} S(t_2 - s)Bu(s)ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \\ &\leq C(t_2) \lim_{t_1 \rightarrow t_2^-} \int_{t_1}^{t_2} |u(s)|_{L_{\mathcal{F}_{t_2}}^2(\Omega;U)} ds = 0. \end{aligned} \quad (2.11)$$

Clearly,

$$\left| \int_{t_1}^{t_2} S(t_2 - s)g(s)dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega;H)} \leq |g|_{L_{\mathbb{F}}^2(t_1, t_2; H)}.$$

Since

$$\left( \int_{t_1}^{t_2} |g(s)|_H^2 ds \right) (\omega) \leq \left( \int_0^{t_2} |g(s)|_H^2 ds \right) (\omega), \quad \mathbb{P}\text{-a.s.},$$

utilizing Lebesgue's dominated convergence theorem, we find that

$$\begin{aligned} \lim_{t_1 \rightarrow t_2^-} \left| \int_{t_1}^{t_2} S(t_2 - s) g(s) dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} &\leq C \lim_{t_1 \rightarrow t_2^-} \mathbb{E} \left( \int_{t_1}^{t_2} |g(s)|_H^2 ds \right) \\ &= C \mathbb{E} \left( \lim_{t_1 \rightarrow t_2^-} \int_{t_1}^{t_2} |g(s)|_H^2 ds \right) = 0. \end{aligned} \quad (2.12)$$

Further, since

$$| [S(t_2 - t_1) - I] f(s) |_H \leq C |f(s)|_H,$$

using Lebesgue's dominated convergence theorem again, we find that

$$\begin{aligned} \lim_{t_1 \rightarrow t_2^-} \left| \int_0^{t_1} S(t_1 - s) [S(t_2 - t_1) - I] f(s) ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} \\ \leq C \lim_{t_1 \rightarrow t_2^-} \int_0^{t_2} | [S(t_2 - t_1) - I] f(s) |_H^2 ds = 0. \end{aligned} \quad (2.13)$$

Similarly, thanks to

$$\left| \int_0^{t_1} S(t_1 - s) Bu(s) ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} \leq C |u|_{L_{\mathbb{F}}^2(0, T; U)},$$

it follows from Lebesgue's dominated convergence theorem that

$$\lim_{t_1 \rightarrow t_2^-} \left| [S(t_2 - t_1) - I] \int_0^{t_1} S(t_1 - s) Bu(s) ds \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} = 0. \quad (2.14)$$

Next,

$$\begin{aligned} &\left| \int_0^{t_1} S(t_1 - s) [S(t_2 - t_1) - I] g(s) dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} \\ &\leq C \mathbb{E} \left( \int_0^{t_2} |S(t_1 - s) [S(t_2 - t_1) - I] g(s)|_H^2 ds \right) \\ &\leq C \mathbb{E} \left( \int_0^{t_2} | [S(t_2 - t_1) - I] g(s) |_H^2 ds \right). \end{aligned} \quad (2.15)$$

It is clear that

$$\int_0^{t_2} | [S(t_2 - t_1) - I] g(s) |_H^2 ds \leq C \int_0^{t_2} |g(s)|_H^2 ds.$$

Then, Lebesgue's dominated convergence theorem together with (2.15), implies that

$$\begin{aligned} \lim_{t_1 \rightarrow t_2^-} \left| \int_0^{t_1} S(t_1 - s) [S(t_2 - t_1) - I] g(s) dW(s) \right|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} \\ \leq C \lim_{t_1 \rightarrow t_2^-} \mathbb{E} \left( \int_0^{t_2} | [S(t_2 - t_1) - I] g(s) |_H^2 ds \right) = 0. \end{aligned} \quad (2.16)$$

From (2.8)–(2.13) and (2.16), we obtain that

$$\lim_{t_1 \rightarrow t_2^-} |Y(t_2) - Y(t_1)|_{L_{\mathcal{F}_{t_2}}^2(\Omega; H)} = 0.$$



Similarly, we can get that

$$\lim_{t_2 \rightarrow t_1^+} |Y(t_2) - Y(t_1)|_{L^2_{\mathcal{F}_{t_2}}(\Omega; H)} = 0.$$

Thus, we prove that  $Y(\cdot) \in C_{\mathbb{F}}([0, +\infty); L^2(\Omega; H))$ .

Fix any  $T_1 \in [0, +\infty)$ . Let us define a map

$$\mathcal{J} : C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H)) \rightarrow C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H))$$

as follows. For any  $X(\cdot) \in C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H))$ ,

$$\begin{aligned} Y(t) = \mathcal{J}(X)(t) &\triangleq S(t)Y_0 + \int_0^t S(t-s)F_1(s)X(s)ds + \int_0^t S(t-s)Bu(s)ds \\ &+ \int_0^t S(t-s)F_2(s)X(s)dW(s). \end{aligned}$$

$\mathcal{J}$  is well-defined following (2.5). We claim that if  $T_1$  is small enough, then  $\mathcal{J}$  is contractive. Indeed, let  $Y_j = \mathcal{J}(X_j)$  ( $j = 1, 2$ ) and  $T_2 > T_1$  be fixed. Then, by (2.7), we can find a constant  $C(T_2) > 0$  such that

$$\begin{aligned} &|Y_1 - Y_2|_{C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H))} \\ &\leq C(T_2)(|F_1X_1 - F_1X_2|_{L^2_{\mathbb{F}}(0, T_1; H)} + |F_2X_1 - F_2X_2|_{L^2_{\mathbb{F}}(0, T_1; H)}) \\ &\leq C(T_2)(|F_1|_{L^\infty_{\mathbb{F}}(0, T_1; H)} + |F_2|_{L^\infty_{\mathbb{F}}(0, T_1; H)})|X_1 - X_2|_{L^2_{\mathbb{F}}(0, T_1; H)} \\ &\leq C(T_2)\sqrt{T_1}(|F_1|_{L^\infty_{\mathbb{F}}(0, T_1; H)} + |F_2|_{L^\infty_{\mathbb{F}}(0, T_1; H)})|X_1 - X_2|_{C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H))}. \end{aligned}$$

Thus, we know that  $\mathcal{J}$  is contractive if  $T_1 < \frac{1}{C(T_2)^2}$ . By means of the Banach fixed point theorem,  $\mathcal{J}$  has a unique fixed point  $Y(\cdot) \in C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H))$ . It is clear that  $Y(\cdot)$  is a mild solution of the following equation:

$$\begin{cases} dY(t) = [AX(t) + F_1(t)X(t)]dt + Bu(t)dt + F_2(t)Y(t)dW(t) & \text{in } (0, T_1], \\ Y(0) = Y_0. \end{cases} \quad (2.17)$$

By (2.7) again, we find that

$$\begin{aligned} &\mathbb{E}|Y(t)|_H^2 \\ &\leq C(T_1)\left(\mathbb{E}|Y_0|_H^2 + |u|_{L^2_{\mathbb{F}}(0, T; U)}^2 + \int_0^t |F_1(s)Y(s)|_H^2 ds + \int_0^t |F_2(s)Y(s)|_H^2 ds\right) \\ &\leq C(T_1)\left[\mathbb{E}|Y_0|_H^2 + |u|_{L^2_{\mathbb{F}}(0, T; U)}^2 + (|F_1|_{L^\infty_{\mathbb{F}}(0, T; H)} + |F_2|_{L^\infty_{\mathbb{F}}(0, T; H)}) \int_0^t |Y(s)|_H^2 ds\right]. \end{aligned}$$

This, together with the Gronwall's inequality, implies that

$$|Y(\cdot)|_{C_{\mathbb{F}}([0, T_1]; L^2(\Omega; H))} \leq C(T_1)[|Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + |u|_{L^2_{\mathbb{F}}(0, T; U)}^2]. \quad (2.18)$$

Repeating the above argument, we obtain the mild solution to the equation (1.2). The uniqueness of such solution is obvious. The desired estimate (2.4) follows from (2.18). This completes the proof of Theorem 2.1.  $\square$

**Remark 2.4** By Theorem 2.1, any

$$(Y_0, u(\cdot)) \in L^2_{\mathcal{F}_0}(\Omega; H) \times L^2_{\mathbb{F}}(0, +\infty; U)$$

uniquely determines the state trajectory  $Y(\cdot; Y_0, u)$  in an explicit way.

As we said before, to apply the Itô formula, we should consider the weak solution to (1.2). Fortunately, we have the following result.

**Proposition 2.4** *The mild solution to (1.2) is also a weak solution to (1.2) and vice versa.*

*Proof:* We first prove that a weak solution to (1.2) is also mild solution.

Assume that  $Y(\cdot)$  is a weak solution to (1.2). For any  $\psi \in D(A^*)$  and  $r \in [0, +\infty)$ , we have that

$$\begin{aligned} \langle Y(r), \psi \rangle_H &= \langle Y_0, \psi \rangle_H + \int_0^r \langle Y(s), A^* \psi \rangle_H ds + \int_0^r \langle F_1(s)Y(s), \psi \rangle_H ds \\ &\quad + \int_0^r \langle u(s), B^* \psi \rangle_H ds + \int_0^r \langle F_2(s)Y(s), \psi \rangle_H dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (2.19)$$

We choose  $\psi = S^*(t-r)A^*\phi$  for some  $\phi \in D((A^*)^2)$  and  $t \in [r, +\infty)$ . From (2.19), we get that

$$\begin{aligned} &\langle Y(r), S^*(t-r)A^*\phi \rangle_H \\ &= \langle Y_0, S^*(t-r)A^*\phi \rangle_H + \int_0^r \langle Y(s), A^*S^*(t-r)A^*\phi \rangle_H ds \\ &\quad + \int_0^r \langle F_1(s)Y(s), S^*(t-r)A^*\phi \rangle_H ds + \int_0^r \langle u(s), B^*S^*(t-r)A^*\phi \rangle_H ds \\ &\quad + \int_0^r \langle F_2(s)Y(s), S^*(t-r)A^*\phi \rangle_H dW(s). \end{aligned} \quad (2.20)$$

Integrating (2.20) with respect to  $r$  from 0 to  $t$ , we obtain that

$$\begin{aligned} &\int_0^t \langle Y(r), S^*(t-r)A^*\phi \rangle_H dr \\ &= \int_0^t \langle Y_0, S^*(t-r)A^*\phi \rangle_H dr + \int_0^t \int_0^r \langle Y(s), A^*S^*(t-r)A^*\phi \rangle_H ds dr \\ &\quad + \int_0^t \int_0^r \langle F_1(s)Y(s), S^*(t-r)A^*\phi \rangle_H ds dr + \int_0^t \int_0^r \langle u(s), B^*S^*(t-r)A^*\phi \rangle_H ds dr \\ &\quad + \int_0^t \int_0^r \langle F_2(s)Y(s), S^*(t-r)A^*\phi \rangle_H dW(s) dr. \end{aligned} \quad (2.21)$$

By a direct computation, we see that

$$\int_0^t \langle Y_0, S^*(t-r)A^*\phi \rangle_H dr = \int_0^t \langle AS(t-r)Y_0, \phi \rangle_H dr = \langle S(t)Y_0, \phi \rangle_H - \langle Y_0, \phi \rangle_H. \quad (2.22)$$

By Fubini's theorem, we find that

$$\begin{aligned} &\int_0^t \int_0^r \langle Y(s), A^*S^*(t-r)A^*\phi \rangle_H ds dr \\ &= \int_0^t \int_s^t \langle Y(s), A^*S^*(t-r)A^*\phi \rangle_H dr ds \\ &= \int_0^t \langle Y(s), S^*(t-s)A^*\phi \rangle_H ds - \int_0^t \langle Y(s), A^*\phi \rangle_H ds, \end{aligned} \quad (2.23)$$

$$\begin{aligned}
& \int_0^t \int_0^r \langle F_1(s)Y(s), S^*(t-r)A^*\phi \rangle_H ds dr \\
&= \int_0^t \int_s^t \langle F_1(s)Y(s), S^*(t-r)A^*\phi \rangle_H dr ds \\
&= \int_0^t \langle F_1(s)Y(s), S^*(t-s)\phi \rangle_H ds - \int_0^t \langle F_1(s)Y(s), \phi \rangle_H ds,
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
& \int_0^t \int_0^r \langle u(s), B^*S^*(t-r)A^*\phi \rangle_H ds dr \\
&= \int_0^t \int_s^t \langle u(s), B^*S^*(t-r)A^*\phi \rangle_H dr ds \\
&= \int_0^t \langle u(s), B^*S^*(t-s)\phi \rangle_H ds - \int_0^t \langle u(s), B^*\phi \rangle_H ds.
\end{aligned} \tag{2.25}$$

By the stochastic Fubini theorem, we obtain that

$$\begin{aligned}
& \int_0^t \int_0^r \langle F_2(s)Y(s), S^*(t-r)A^*\phi \rangle_H dW(s) dr \\
&= \int_0^t \int_s^t \langle F_2(s)Y(s), S^*(t-r)A^*\phi \rangle_H dr dW(s) \\
&= \int_0^t \langle F_2(s)Y(s), S^*(t-s)\phi \rangle_H dW(s) - \int_0^t \langle F_2(s)Y(s), \phi \rangle_H dW(s).
\end{aligned} \tag{2.26}$$

From (2.21)–(2.26), we end up with

$$\begin{aligned}
& \left\langle Y(t) - S(t)Y_0 - \int_0^t S(t-s)F_1(s)Y(s)ds - \int_0^t S(t-s)Bu(s)ds \right. \\
& \quad \left. - \int_0^t S(t-s)F_2(s)Y(s)dW(s), \phi \right\rangle_H = 0.
\end{aligned} \tag{2.27}$$

Since  $D((A^*)^2)$  is dense in  $H$ , we know that (2.27) also holds for any  $\phi \in H$ , which concludes that  $Y$  is a mild solution to (1.2).

Next, we prove that a mild solution to (1.2) is also a weak solution.

Assume that  $Y$  is a mild solution to (1.2). Then for any  $\psi \in D(A^*)$  and  $r \in [0, +\infty)$ , we have that

$$\begin{aligned}
& \left\langle Y(r) - S(r)Y_0 - \int_0^r S(r-s)F_1(s)Y(s)ds - \int_0^r S(r-s)Bu(s)ds \right. \\
& \quad \left. - \int_0^r S(r-s)F_2(s)Y(s)dW(s), A^*\psi \right\rangle_H = 0.
\end{aligned} \tag{2.28}$$

Integrating (2.28) from 0 to  $t$  with respect to  $r$ , we find that

$$\begin{aligned}
& \int_0^t \langle Y(r), A^*\psi \rangle_H dr \\
&= \int_0^t \langle S(r)Y_0, A^*\psi \rangle_H dr + \int_0^t \int_0^r \langle S(r-s)F_1(s)Y(s), A^*\psi \rangle_H ds dr \\
& \quad + \int_0^t \int_0^r \langle S(r-s)Bu(s), A^*\psi \rangle_H ds dr + \int_0^t \int_0^r \langle S(r-s)F_2(s)Y(s), A^*\psi \rangle_H dW(s) dr.
\end{aligned} \tag{2.29}$$

First, we have that

$$\int_0^t \langle S(r)Y_0, A^*\psi \rangle_H dr = \int_0^t \langle AS(r)Y_0, \psi \rangle_H dr = \langle S(t)Y_0, \psi \rangle_H - \langle Y_0, \psi \rangle_H. \quad (2.30)$$

By Fubini's theorem, we get that

$$\begin{aligned} & \int_0^t \int_0^r \langle S(r-s)F_1(s)Y(s), A^*\psi \rangle_H ds dr \\ &= \int_0^t \int_s^t \langle AS(r-s)F_1(s)Y(s), \psi \rangle_H dr ds \\ &= \int_0^t \langle S(t-s)F_1(s)Y(s), \psi \rangle_H ds - \int_0^t \langle F_1(s)Y(s), \psi \rangle_H ds \end{aligned} \quad (2.31)$$

and

$$\begin{aligned} & \int_0^t \int_0^r \langle S(r-s)Bu(s), A^*\psi \rangle_H ds dr \\ &= \int_0^t \int_s^t \langle AS(r-s)Bu(s), \psi \rangle_H dr ds \\ &= \int_0^t \langle S(t-s)Bu(s), \psi \rangle_H ds - \int_0^t \langle u(s), B^*\psi \rangle_H ds. \end{aligned} \quad (2.32)$$

Thanks to the stochastic Fubini theorem, we obtain that

$$\begin{aligned} & \int_0^t \int_0^r \langle S(r-s)F_2(s)Y(s), A^*\psi \rangle_H dW(s) dr \\ &= \int_0^t \int_s^t \langle AS(r-s)F_2(s)Y(s), \psi \rangle_H dW(s) dr \\ &= \int_0^t \langle S(t-s)F_2(s)Y(s), \psi \rangle_H dW(s) - \int_0^t \langle F_2(s)Y(s), \psi \rangle_H dW(s). \end{aligned} \quad (2.33)$$

From (2.29)–(2.33), we have that

$$\begin{aligned} \langle Y(t), \psi \rangle_H &= \langle Y_0, \psi \rangle_H + \int_0^t \langle Y(s), A^*\psi \rangle_H ds + \int_0^t \langle F_1(s)Y(s), \psi \rangle_H ds \\ &\quad + \int_0^t \langle u(s), B^*\psi \rangle_H ds + \int_0^t \langle F_2(s)Y(s), \psi \rangle_H dW(s), \end{aligned} \quad (2.34)$$

which shows that  $Y$  is a weak solution to (1.2).  $\square$

### 2.3 Admissible observation operator

In this subsection, we study the admissible observation operator. Let us first recall its definition.

Define a family of operators  $\{\Psi_t\}_{t \geq 0}$  from  $D(A)$  to  $L^2(0, +\infty; \tilde{U})$  as follows:

$$(\Psi_t \eta)(s) = \begin{cases} \mathcal{C}S(s)\eta, & \text{if } s \in [0, t], \\ 0, & \text{if } s \in (t, +\infty). \end{cases} \quad (2.35)$$

**Definition 2.2** *The operator  $\mathcal{C} \in \mathcal{L}(D(A), \tilde{U})$  is called an admissible observation operator for  $\{S(t)\}_{t \geq 0}$  if for some  $t_0 > 0$ ,  $\Psi_{t_0}$  has a continuous extension to  $H$ .*

Although the definition of the admissible observation operator comes from the investigation of the deterministic control system, we will show that it is also a suitable notion in the study of stochastic control system.

We first recall the following result.

**Proposition 2.5** *If  $\mathcal{C} \in \mathcal{L}(D(A), \tilde{U})$  is admissible, then for every  $t \geq 0$ ,  $\Psi_t \in \mathcal{L}(H, L^2(0, +\infty; \tilde{U}))$ .*

**Proposition 2.6**  *$\mathcal{C} \in \mathcal{L}(D(A), \tilde{U})$  is admissible if and only if for every  $t > 0$ , there is a constant  $C = C(t) > 0$  such that*

$$\int_0^t |\mathcal{C}S(s)\eta|_{\tilde{U}}^2 ds \leq C(t)|\eta|_H^2, \quad \forall \eta \in H.$$

We refer the readers to [26] for the proofs of Propositions 2.5 and 2.6.

In the stochastic context, we have to consider the effect from the noise, i.e., the term  $\int_0^s S(s-r)Y(r)dW(r)$  and the fact that the state space  $L^2_{\mathcal{F}_t}(\Omega; H)$  depends on  $t$ . Fortunately, we have the following result.

**Proposition 2.7** *Let  $\mathcal{C} \in \mathcal{L}(D(A), \tilde{U})$  be an admissible observation operator for  $\{S(t)\}_{t \geq 0}$ . Then for any  $t > 0$ , there exists a constant  $C(t) > 0$  such that for any  $Y_0 \in L^2_{\mathcal{F}_0}(\Omega; H)$ , the solution to (1.2) with  $u = 0$  satisfies that*

$$\mathbb{E} \int_0^t |\mathcal{C}Y(s)|_{\tilde{U}}^2 ds \leq C(t)\mathbb{E}|Y_0|_H^2. \quad (2.36)$$

**Remark 2.5** *If  $Y(\cdot)$  is a solution to a stochastic PDE and  $\mathcal{C}$  is a boundary observation operator, inequalities in the form of (2.36) are usually called the hidden regularity of the solution, i.e., it does not follow directly from the classical trace theorem of Sobolev space. We refer the readers to [14, 16, 17, 31] for the hidden regularity for some stochastic PDEs.*

*Proof of Proposition 2.7:* By the closed-graph theorem, it is an easy matter to see that there is a constant  $C = C(t) > 0$  such that

$$\int_0^t |\mathcal{C}S(r)\eta|_H^2 dr \leq C(t)|\eta|_H^2. \quad (2.37)$$

Since

$$Y(s) = S(s)Y_0 + \int_0^s S(s-r)F_1(r)Y(r)dr + \int_0^s S(s-r)F_2(r)Y(r)dW(r),$$

we have that

$$\begin{aligned} & \mathbb{E} \int_0^t |\mathcal{C}Y(s)|_{\tilde{U}}^2 ds \\ &= \mathbb{E} \int_0^t \left| S(s)Y_0 + \int_0^s S(s-r)F_1(r)Y(r)dr + \int_0^s S(s-r)F_2(r)Y(r)dW(r) \right|_{\tilde{U}}^2 ds \\ &\leq 3\mathbb{E} \int_0^t |\mathcal{C}S(s)Y_0|_{\tilde{U}}^2 ds + 3\mathbb{E} \int_0^t \left| \mathcal{C} \int_0^s S(s-r)F_1(r)Y(r)dr \right|_{\tilde{U}}^2 ds \\ &\quad + 3\mathbb{E} \int_0^t \left| \mathcal{C} \int_0^s S(s-r)F_2(r)Y(r)dW(r) \right|_{\tilde{U}}^2 ds \\ &\leq C(t) \left( \mathbb{E} \int_0^t |\mathcal{C}S(s)Y_0|_{\tilde{U}}^2 ds + \mathbb{E} \int_0^t |\mathcal{C}S(s-r)F_1(r)Y(r)|_{\tilde{U}}^2 ds + \mathbb{E} \int_0^t |\mathcal{C}S(s-r)F_2(r)Y(r)|_{\tilde{U}}^2 ds \right). \end{aligned}$$

This, together with Proposition 2.6, implies that

$$\begin{aligned}
& \mathbb{E} \int_0^t |\mathcal{C}Y(s)|_{\tilde{U}}^2 ds \\
& \leq C(t) \left( \mathbb{E} |Y_0|_H^2 ds + \mathbb{E} \int_0^t |F_1(s)Y(s)|_H^2 ds + \mathbb{E} \int_0^t |F_2(s)Y(s)|_H^2 ds \right) \\
& \leq C(t) \mathbb{E} |Y_0|_H^2.
\end{aligned}$$

□

**Remark 2.6** Obviously, every  $\mathcal{C} \in \mathcal{L}(H, \tilde{U})$  is admissible for  $\{S(t)\}_{t \geq 0}$ . If  $\mathcal{C}$  is an admissible observation operator for  $\{S(t)\}_{t \geq 0}$ , then we denote the (unique) extension of  $\Psi_t$  to  $L_{\mathcal{F}_0}^2(\Omega; H)$  by the same symbol.

## 2.4 Stochastic well-posed linear system

We begin with the definition of the stochastic well-posed linear system.

**Definition 2.3** Let  $B \in \mathcal{L}(U; H_{-1})$  be an admissible control operator and  $\mathcal{C} \in \mathcal{L}(D(A), \tilde{U})$  be an admissible observation operator. We say (1.2) is well-posed if there is a  $t_0 > 0$  and a  $C(t_0) > 0$  such that for any  $Y_0 \in L_{\mathcal{F}_0}^2(\Omega; H)$  and  $u \in L_{\mathbb{F}}^2(0, +\infty; U)$ ,

$$|\mathcal{C}Y|_{L_{\mathbb{F}}^2(0, t_0; \tilde{U})} \leq C(t_0) (|Y_0|_{L_{\mathcal{F}_0}^2(\Omega; H)} + |u|_{L_{\mathbb{F}}^2(0, t_0; U)}). \quad (2.38)$$

**Remark 2.7** Although our definition of stochastic well-posed linear system seems different from the classical definition to the deterministic well-posed linear system, the spirit is the same, i.e., both the control and observation operators are admissible and the map from the input to the output is bounded.

Similar to Propositions 2.2 and 2.5, we have the following result.

**Proposition 2.8** If the system (1.2) is well-posed, then for any  $t > 0$ , there is a constant  $C(t) > 0$  such that for any  $Y_0 \in L_{\mathcal{F}_0}^2(\Omega; H)$  and  $u \in L_{\mathbb{F}}^2(0, +\infty; U)$ ,

$$|\mathcal{C}Y|_{L_{\mathbb{F}}^2(0, t; \tilde{U})} \leq C(t) (|Y_0|_{L_{\mathcal{F}_0}^2(\Omega; H)} + |u|_{L_{\mathbb{F}}^2(0, t; U)}). \quad (2.39)$$

*Proof:* For any  $u \in L_{\mathbb{F}}^2(0, +\infty; U)$ , by the uniqueness of the solution to (1.2), we have that

$$\chi_{[0, 2t_0]} Y(\cdot; 0, u) = \chi_{[0, t_0]} Y(\cdot; Y_0, \chi_{[0, t_0]} u) + \chi_{[t_0, 2t_0]} Y(\cdot; Y(t_0; Y_0, \chi_{[0, t_0]} u), \chi_{[t_0, 2t_0]} u).$$

Hence, we see that

$$\begin{aligned}
& |\mathcal{C}Y|_{L_{\mathbb{F}}^2(0, 2t_0; \tilde{U})} \\
& \leq |\mathcal{C}[\chi_{[0, t_0]} Y(\cdot; Y_0, \chi_{[0, t_0]} u)]|_{L_{\mathbb{F}}^2(0, t_0; \tilde{U})} + |\mathcal{C}[\chi_{[t_0, 2t_0]} Y(\cdot; Y(t_0; Y_0, \chi_{[0, t_0]} u), \chi_{[t_0, 2t_0]} u)]|_{L_{\mathbb{F}}^2(t_0, 2t_0; \tilde{U})}.
\end{aligned} \quad (2.40)$$

From (2.38), we have that

$$|\mathcal{C}[\chi_{[0, t_0]} Y(\cdot; Y_0, \chi_{[0, t_0]} u)]|_{L_{\mathbb{F}}^2(0, t_0; \tilde{U})} \leq C(t_0) (|Y_0|_{L_{\mathcal{F}_0}^2(\Omega; H)} + |u|_{L_{\mathbb{F}}^2(0, t_0; U)}). \quad (2.41)$$

Next, put  $\widehat{Y}(t) = Y(t + t_0; Y(t_0; Y_0, \chi_{[0, t_0]}u), \chi_{[t_0, 2t_0]}u)$ . Then we have that (see [20, Chapter 4] for the details)

$$\begin{aligned}
& |\mathcal{C}(\chi_{[t_0, 2t_0]}\widehat{Y})|_{L^2_{\mathbb{F}}(t_0, 2t_0; \widetilde{U})}^2 \\
&= \mathbb{E} \left| \mathcal{C}S(s)\widehat{Y}(t_0) + \mathcal{C} \int_0^{t_0} S(t_0 - s)F_1(s)\widehat{Y}(s)ds + \mathcal{C} \int_0^{t_0} S(t_0 - s)Bu(s + t_0)ds \right. \\
&\quad \left. + \mathcal{C} \int_0^{t_0} S(t_0 - s)F_2(s)\widehat{Y}(s)d[W(s + t_0) - W(t_0)] \right|_{\widetilde{U}}^2 \\
&\leq 2\mathbb{E} \left| \mathcal{C}S(s)\widehat{Y}(t_0) + \mathcal{C} \int_0^{t_0} S(t_0 - s)F_1(s)\widehat{Y}(s)ds + \mathcal{C} \int_0^{t_0} S(t_0 - s)F_2(s)\widehat{Y}(s)d[W(s + t_0) - W(t_0)] \right|_{\widetilde{U}}^2 \\
&\quad + 2\mathbb{E} \left| \mathcal{C} \int_0^{t_0} S(t_0 - s)Bu(s + t_0)ds \right|_{\widetilde{U}}^2.
\end{aligned} \tag{2.42}$$

Since  $\mathcal{C}$  is an admissible observation operator, we know that

$$\begin{aligned}
& 2\mathbb{E} \left| \mathcal{C}S(s)\widehat{Y}(t_0) + \mathcal{C} \int_0^{t_0} S(t_0 - s)F_1(s)\widehat{Y}(s)ds + \mathcal{C} \int_0^{t_0} S(t_0 - s)F_2(s)\widehat{Y}(s)d[W(s + t_0) - W(t_0)] \right|_{\widetilde{U}}^2 \\
&\leq C \left( \mathbb{E}|\widehat{Y}(t_0)|_H^2 + \mathbb{E} \int_0^{t_0} |\widehat{Y}(s)|_H^2 ds \right).
\end{aligned} \tag{2.43}$$

By Choosing  $Y_0 = 0$  and noting that the system (1.2) is well-posed, from (2.38), we get that

$$\mathbb{E} \left| \mathcal{C} \int_0^{t_0} S(t_0 - s)Bu(s)ds \right|_{\widetilde{U}}^2 \leq C(t_0)|u|_{L^2_{\mathbb{F}}(0, t_0; U)}^2. \tag{2.44}$$

Hence, we have that for any  $u \in L^2_{B^r}(0, +\infty; U) \subset L^2_{\mathbb{F}}(0, +\infty; U)$ ,

$$\begin{aligned}
\left| \mathcal{C} \int_0^{t_0} S(t_0 - s)Bu(s)ds \right|_{\widetilde{U}}^2 &= \mathbb{E} \left| \mathcal{C} \int_0^{t_0} S(t_0 - s)Bu(s)ds \right|_{\widetilde{U}}^2 \\
&\leq C(t_0) \mathbb{E} \int_0^{t_0} |u(s)|_{\widetilde{U}}^2 ds \\
&= C(t_0) \int_0^{t_0} |u(s)|_{\widetilde{U}}^2 ds.
\end{aligned} \tag{2.45}$$

From (2.45), we obtain that

$$\mathbb{E} \left| \mathcal{C} \int_0^{t_0} S(t_0 - s)Bu(s + t_0)ds \right|_{\widetilde{U}}^2 \leq C(t_0) \mathbb{E} \int_0^{t_0} |u(s + t_0)|_{\widetilde{U}}^2 ds = C(t_0)|u|_{L^2_{\mathbb{F}}(t_0, 2t_0; U)}^2. \tag{2.46}$$

According to (2.40)–(2.43) and (2.46), we conclude that

$$|\mathcal{C}Y|_{L^2_{\mathbb{F}}(0, 2t_0; \widetilde{U})} \leq C(2t_0)(|u|_{L^2_{\mathbb{F}}(0, 2t_0; U)} + |Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)}). \tag{2.47}$$

By induction, we can prove that for any  $n \in \mathbb{N}$ , there is a constant  $C(n) > 0$  such that for any  $Y_0 \in L^2_{L^2_{\mathcal{F}_0}(\Omega; H)}$  and  $u \in L^2_{\mathbb{F}}(0, +\infty; U)$ ,

$$|\mathcal{C}Y|_{L^2_{\mathbb{F}}(0, 2^n t_0; \widetilde{U})} \leq C(n)(|u|_{L^2_{\mathbb{F}}(0, 2^n t_0; U)} + |Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)}). \tag{2.48}$$

For any  $t > 0$ , there is a  $n \in \mathbb{N}$  such that  $0 < t < 2^n t_0$ . For a given  $u \in L^2_{\mathbb{F}}(0, +\infty; U)$ , put

$$\tilde{u}(s) = \begin{cases} u(s), & \text{if } s \in [0, t], \\ 0, & \text{if } s \in (t, +\infty). \end{cases}$$

According to (2.48), we get that

$$\begin{aligned} |CY|_{L^2_{\mathbb{F}}(0, t; \tilde{U})} &\leq C(n) (|\tilde{u}|_{L^2_{\mathbb{F}}(0, 2^n t_0; U)} + |Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)}) \\ &\leq C(n) (|u|_{L^2_{\mathbb{F}}(0, t; U)} + |Y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)}). \end{aligned} \quad (2.49)$$

□

### 3 Well-posedness of controlled stochastic heat equation

In this section, we study a stochastic heat equation with boundary control and observation. We show that it is a stochastic well-posed linear system.

Let  $G \subset \mathbb{R}^d (d \in \mathbb{N})$  be a bounded domain with the  $C^2$  boundary  $\Gamma$ . Consider the following stochastic heat equation:

$$\begin{cases} dy - \Delta y dt = ay dt + by dW(t) & \text{in } G \times (0, +\infty), \\ \frac{\partial y}{\partial \nu} = u & \text{on } \Gamma \times (0, +\infty), \\ y(0) = y_0 & \text{in } G, \\ z = y & \text{on } \Gamma \times (0, +\infty). \end{cases} \quad (3.1)$$

Here  $a, b \in L^\infty_{\mathbb{F}}(0, +\infty; L^\infty(G))$  and  $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$ .

**Theorem 3.1** *With the choice that  $H = L^2(G)$ ,  $U = H^{-\frac{1}{2}}(\Gamma)$  and  $\tilde{U} = H^{\frac{1}{2}}(\Gamma)$ , the system (3.1) is well-posed, i.e., for any  $T > 0$ , there is a constant  $C = C(T) > 0$  such that for any  $y_0 \in L^2_{\mathcal{F}_0}(\Omega; H)$  and  $u \in L^2_{\mathbb{F}}(0, T; U)$ , there is a unique mild (also weak) solution  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$  to (3.1) such that*

$$|y|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; H))} + |z|_{L^2_{\mathbb{F}}(0, T; \tilde{U})} \leq C(T) (|y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + |u|_{L^2_{\mathbb{F}}(0, T; U)}). \quad (3.2)$$

*Proof:* We divide the proof into three steps.

**Step 1.** We first handle the case that  $u \in C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-\frac{1}{2}}(\Gamma)))$ .

Consider the following elliptic equation:

$$\begin{cases} \Delta v(t, \omega) = 0 & \text{in } G, \\ \frac{\partial v(t, \omega)}{\partial \nu} = u(t, \omega) & \text{on } \Gamma_0. \end{cases}$$

We claim that

$$v \in C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^1(G))). \quad (3.3)$$

Indeed, from the classical theory of elliptic equations with Neumann boundary condition (see [10, Chapter 3] for example), we have that

$$|v|_{H^1(G)}^2 \leq C |u|_{H^{-\frac{1}{2}}(\Gamma)}^2, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (3.4)$$



This, together with the fact that  $u(t) \in L^2_{\mathcal{F}_t}(\Omega; H^{-\frac{1}{2}}(\Gamma))$ , implies that  $v(t) \in L^2_{\mathcal{F}_t}(\Omega; H^1(G))$ . Furthermore, it follows from (3.4) that

$$\mathbb{E}|v(s) - v(t)|^2_{H^1(G)} \leq C\mathbb{E}|u(s) - u(t)|^2_{H^{-\frac{1}{2}}(\Gamma)}. \quad (3.5)$$

Since  $u \in C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-\frac{1}{2}}(\Gamma)))$ , we find from (3.5) that  $v \in C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^1(\Gamma)))$ .

Consider the following stochastic heat equation:

$$\begin{cases} d\tilde{y} - \Delta\tilde{y}dt = a\tilde{y}dt + (av - v_t)dt + b\tilde{y}dW(t) + bvdW(t) & \text{in } G \times (0, +\infty), \\ \frac{\partial\tilde{y}}{\partial\nu} = 0 & \text{on } \Gamma \times (0, +\infty), \\ \tilde{y}(0) = y_0 - v(0) & \text{in } G. \end{cases} \quad (3.6)$$

According to the classical well-posedness result for stochastic heat equation (see [3, Chapter 6] for example), we know that (3.6) admits a unique mild solution  $\tilde{y} \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \cap L^2_{\mathbb{F}}(0, T; H^1(G))$ , which is also a weak solution to (3.6). Let  $y = \tilde{y} + v$ . Then, it is easy to see that  $y$  is a solution to (3.1).

**Step 2.** In this step, we establish an energy estimate for the solution to (3.1).

By Itô's formula, we have that

$$\begin{aligned} & \mathbb{E} \int_G |y(t)|^2 dx + \mathbb{E} \int_0^t \int_G |\nabla y|^2 dx ds \\ &= \mathbb{E} \int_G |y(0)|^2 dx + 2\mathbb{E} \int_0^t \langle u, y \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} ds + 2\mathbb{E} \int_0^t \int_G ay^2 dx ds + \mathbb{E} \int_0^t \int_G b^2 y^2 dx ds. \end{aligned} \quad (3.7)$$

Thanks to the classical trace theorem in Sobolev space, we get that

$$\begin{aligned} & \mathbb{E} \int_0^t \langle u, y \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} ds \\ & \leq \mathbb{E} \int_0^t |u|_{H^{-\frac{1}{2}}(\Gamma)} |y|_{H^{\frac{1}{2}}(\Gamma)} ds \leq C\mathbb{E} \int_0^t |u|_{H^{-\frac{1}{2}}(\Gamma)} |y|_{H^1(G)} ds \\ & \leq 2C\mathbb{E} \int_0^t |u|^2_{H^{-\frac{1}{2}}(\Gamma)} ds + \frac{1}{4}\mathbb{E} \int_0^t |y|^2_{H^1(G)} ds \\ & \leq 2C\mathbb{E} \int_0^t |u|^2_{H^{-\frac{1}{2}}(\Gamma)} ds + \frac{1}{4}\mathbb{E} \int_0^t (|\nabla y|^2 + y^2) dx ds. \end{aligned}$$

This, together with (3.7), implies that

$$\begin{aligned} & \mathbb{E} \int_G |y(t)|^2 dx + \frac{1}{2}\mathbb{E} \int_0^t \int_G |\nabla y|^2 dx ds \\ & \leq \mathbb{E} \int_G |y(0)|^2 dx + 4C\mathbb{E} \int_0^t |u|^2_{H^{-\frac{1}{2}}(\Gamma)} ds + \mathbb{E} \int_0^t \int_G (2a + b^2 + 1)y^2 dx ds \\ & \leq \mathbb{E} \int_G |y(0)|^2 dx + 4C\mathbb{E} \int_0^t |u|^2_{H^{-\frac{1}{2}}(\Gamma)} ds + (2|a|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(G))} + |b|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(G))}^2 + 1)\mathbb{E} \int_0^t \int_G y^2 dx ds. \end{aligned} \quad (3.8)$$

It follows from Gronwall's inequality and (3.8) that

$$\mathbb{E} \int_G |y(t)|^2 dx + \mathbb{E} \int_0^T \int_G |\nabla y|^2 dx ds \leq C \left( \mathbb{E} \int_G |y(0)|^2 dx + \mathbb{E} \int_0^T |u|^2_{H^{-\frac{1}{2}}(\Gamma)} ds \right). \quad (3.9)$$

**Step 3.** In this step, let us deal with the case that  $u \in L^2_{\mathbb{F}}(0, T; H^{-\frac{1}{2}}(\Gamma))$ .

We can find a sequence  $\{u_n\}_{n=1}^{\infty} \subset C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-\frac{1}{2}}(\Gamma)))$  such that

$$\lim_{n \rightarrow \infty} u_n = u \quad \text{in } L^2_{\mathbb{F}}(0, T; H^{-\frac{1}{2}}(\Gamma)). \quad (3.10)$$

Denote by  $y_n$  the solution to (3.1) with the initial datum  $y_0$  and the control  $u_n$ . From (3.9) and (3.10), we know that  $\{y_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times L^2_{\mathbb{F}}(0, T; H^1(G))$ . Thus, there is a unique  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times L^2_{\mathbb{F}}(0, T; H^1(G))$  such that

$$\lim_{n \rightarrow \infty} y_n = y \quad \text{in } C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times L^2_{\mathbb{F}}(0, T; H^1(G)). \quad (3.11)$$

From the definition of  $y_n$ , we have that for any  $t \in [0, T]$  and  $\eta \in H^1(G)$ ,

$$\begin{aligned} & \int_G y_n(t) \eta dx - \int_G y_n(0) \eta dx - \int_0^t \langle u_n, \eta \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} ds + \int_0^t \int_G \nabla y_n \nabla \eta dx ds \\ &= \int_0^t \int_G a y_n \nabla \eta dx ds + \int_0^t \int_G a y_n \nabla \eta dx dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.12)$$

Thanks to (3.11) and (3.12), we conclude that for any  $t \in [0, T]$ ,

$$\begin{aligned} & \int_G y(t) \eta dx - \int_G y(0) \eta dx - \int_0^t \langle u, \eta \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} ds + \int_0^t \int_G \nabla y \nabla \eta dx ds \\ &= \int_0^t \int_G a y \nabla \eta dx ds + \int_0^t \int_G a y \nabla \eta dx dW(s), \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.13)$$

Hence,  $y$  is a weak solution (also a mild solution) to (3.1) and satisfies (3.2).  $\square$

## 4 Well-posedness of controlled stochastic Schrödinger equation

This section is devoted to the study of a stochastic Schrödinger equation with boundary control and observation. We show that it is a stochastic well-posed linear system.

In this section, we assume that  $\mathbb{F}$  is the natural filtration generated by the Brownian motion  $\{W(t)\}_{t \geq 0}$ . Consider the following stochastic Schrödinger equation:

$$\begin{cases} dy + i\Delta y dt = a y dt + b y dW(t) & \text{in } G \times (0, +\infty), \\ y = u & \text{on } \Gamma \times (0, +\infty), \\ y(0) = y_0 & \text{in } G, \\ \varphi = -i \frac{\partial(-\Delta)^{-1} y}{\partial \nu} & \text{on } \Gamma \times (0, +\infty). \end{cases} \quad (4.1)$$

Here  $y_0 \in H^{-1}(G)$  and  $a, b \in L^{\infty}_{\mathbb{F}}(0, +\infty; W^{1,+\infty}_0(G))$ .

Let  $H = H^{-1}(G)$  and  $U = \tilde{U} = L^2(\Gamma_0)$ . We have the following result:

**Theorem 4.1** *System (4.1) is well-posed, i.e., for any  $T > 0$ , there is a constant  $C = C(T) > 0$  such that for any  $y_0 \in L^2_{\mathcal{F}_0}(\Omega; H)$  and  $u \in L^2_{\mathbb{F}}(0, T; U)$ , there is a unique solution  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$  to (4.1) such that*

$$|y|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; H))} + |z|_{L^2_{\mathbb{F}}(0, T; \tilde{U})} \leq C(T) (|y_0|_{L^2_{\mathcal{F}_0}(\Omega; H)} + |u|_{L^2_{\mathbb{F}}(0, T; U)}). \quad (4.2)$$

**Remark 4.1** *One can also consider stochastic Schrödinger equations with variable coefficients. Following the method in [5] and the proof of Theorem 4.1, one can see that Theorem 4.1 is also true if the Laplacian operator in (4.1) is replaced by a general elliptic operator. As we said before, to present the key idea in a simple way, we do not pursue the full technical generality.*

To prove Theorem 4.1, we first write it in an abstract form. To this end, let us define an unbounded linear operator on  $H$  as follows:

$$\begin{cases} D(A) = H_0^1(G), \\ \langle Af, g \rangle_{H^{-1}(G), H_0^1(G)} = \int_G \nabla f(x) \cdot \overline{\nabla g(x)} dx, \quad \forall f, g \in H_0^1(G). \end{cases}$$

Define a map  $\Upsilon : L^2(\Gamma_0) \rightarrow L^2(G)$  as follows:

$$\Upsilon u = v,$$

where  $v$  is the solution to

$$\begin{cases} \Delta v = 0 & \text{in } G, \\ v = u & \text{on } \Gamma. \end{cases}$$

From the definition of  $\Upsilon$  and the classical theory for elliptic equations with non-homogeneous boundary condition (see [11, Chapter 2] for example), we get that there is a constant  $C > 0$  such that for any  $(t, \omega) \in (0, T) \times \Omega$ ,

$$\int_G |i\Upsilon u(t, \omega)|^2 dx \leq C \int_\Gamma |u(t, \omega)|^2 d\Gamma,$$

which deduces that

$$\int_0^T \int_G |i\Upsilon u|^2 dx dt \leq C \int_0^T \int_\Gamma |u|^2 d\Gamma dt. \quad (4.3)$$

Define two operators  $J, K \in \mathcal{L}(L_{\mathbb{F}}^2(0, T; H))$  as

$$Jh = ah, \quad Kh = bh, \quad \forall h \in L_{\mathbb{F}}^2(0, T; H).$$

Then, the system (4.1) can be written as

$$dy - iA(y - \Upsilon u) = Jydt + KydW \quad \text{in } (0, +\infty). \quad (4.4)$$

Clearly,

$$D(A) \subset D(A^{\frac{1}{2}}) \hookrightarrow H \hookrightarrow [D(A)^{\frac{1}{2}}]' \subset [D(A)]'.$$

Denote by  $\tilde{A}$  the extension of  $A$  as a bounded linear operator from  $D(A^{\frac{1}{2}})$  to  $[D(A)^{\frac{1}{2}}]'$  as follows:

$$\langle \tilde{A}f, g \rangle_{[D(A)^{\frac{1}{2}}]', D(A^{\frac{1}{2}})} = \langle A^{\frac{1}{2}}f, A^{\frac{1}{2}}g \rangle_H, \quad \forall f, g \in D(A)^{\frac{1}{2}}.$$

Then  $i\tilde{A}$  generates a  $C_0$ -group on  $[D(A)^{\frac{1}{2}}]'$ . Thus, (4.4) can be reduced to

$$dy = i\tilde{A}y + Budt + Jydt + KydW \quad \text{in } (0, \infty), \quad (4.5)$$

where  $B \in \mathcal{L}(U, [D(A)^{\frac{1}{2}}]')$  such that

$$B\eta = -i\tilde{A}\Upsilon\eta, \quad \forall \eta \in U. \quad (4.6)$$

Denote by  $B^*$  the adjoint operator of  $B$ . Then,  $B^* \in \mathcal{L}(D(A)^{\frac{1}{2}}, U)$  and

$$\langle B^* f, \eta \rangle_U = \langle f, B\eta \rangle_{D(A)^{\frac{1}{2}}, [D(A)^{\frac{1}{2}}]'}, \quad \forall f \in D(A)^{\frac{1}{2}}, \eta \in U.$$

For any  $f \in D(A)$  and  $\eta \in C_0^\infty(\Gamma)$ , we have that

$$\begin{aligned} \langle f, B\eta \rangle_{D(A)^{\frac{1}{2}}, [D(A)^{\frac{1}{2}}]'} &= \langle Af, \tilde{A}^{-1}B\eta \rangle_H = i \langle Af, \Upsilon\eta \rangle_H = i \langle A^{\frac{1}{2}}f, A^{-\frac{1}{2}}\Upsilon\eta \rangle_{L^2(G)} \\ &= i \langle AA^{-1}f, \Upsilon\eta \rangle_{L^2(G)} = - \left\langle i \frac{\partial((-\Delta)^{-1}f)}{\partial\nu}, \eta \right\rangle_U. \end{aligned}$$

Since  $C_0^\infty(\Gamma_0)$  is dense in  $L^2(\Gamma_0)$ , we have that

$$B^* = -i \frac{\partial((-\Delta)^{-1}f)}{\partial\nu} \Big|_{\Gamma_0}. \quad (4.7)$$

The system (4.1) can be written in the following abstract form:

$$\begin{cases} dy = i\tilde{A}ydt + Budy + Jydt + KydW(t) & \text{in } (0, +\infty), \\ y(0) = y_0, \\ \varphi = B^*y & \text{in } (0, +\infty). \end{cases} \quad (4.8)$$

Next, let us give the following result.

**Proposition 4.1** *Let  $\mu = \mu(x) = (\mu^1, \dots, \mu^d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a vector field of class  $C^1$  and  $\hat{\varphi}$  an  $H_{loc}^2(\mathbb{R}^d)$ -valued  $\mathbb{F}$ -adapted semi-martingale. Then for a.e.  $x \in \mathbb{R}^n$  and  $\mathbb{P}$ -a.s.  $\omega \in \Omega$ , it holds that*

$$\begin{aligned} &\mu \cdot \nabla \bar{\varphi}(d\hat{\varphi} + i\Delta\hat{\varphi}dt) - \mu \cdot \nabla \hat{\varphi}(d\bar{\varphi} - i\Delta\bar{\varphi}dt) \\ &= \nabla \cdot \left[ i(\mu \cdot \nabla \bar{\varphi})\nabla \hat{\varphi} + i(\mu \cdot \nabla \hat{\varphi})\nabla \bar{\varphi} - (\hat{\varphi}d\bar{\varphi})\mu - i|\nabla \hat{\varphi}|^2\mu \right] dt + d(\mu \cdot \nabla \bar{\varphi}\hat{\varphi}) \\ &\quad - i \sum_{j,k=1}^d (\mu_j^k + \mu_k^j)\hat{\varphi}_j\bar{\varphi}_kdt + i(\nabla \cdot \mu)|\nabla \hat{\varphi}|^2dt + (\nabla \cdot \mu)\hat{\varphi}d\bar{\varphi} - (\mu \cdot \nabla d\bar{\varphi})d\hat{\varphi}. \end{aligned} \quad (4.9)$$

*Proof of Proposition 4.1 :* The proof is a direct computation. We have that

$$\begin{aligned} &i \sum_{k=1}^d \sum_{j=1}^d \mu^k \bar{\varphi}_k \hat{\varphi}_{jj} + i \sum_{k=1}^d \sum_{j=1}^d \mu^k \hat{\varphi}_k \bar{\varphi}_{jj} \\ &= i \sum_{k=1}^d \sum_{j=1}^d \left[ (\mu^k \bar{\varphi}_k \hat{\varphi}_j)_j + (\mu^k \hat{\varphi}_k \bar{\varphi}_j)_j + \mu_k^k |\hat{\varphi}_j|^2 - (\mu^k |\hat{\varphi}_j|^2)_k - (\mu_j^k + \mu_k^j) \bar{\varphi}_k \hat{\varphi}_j \right] \end{aligned} \quad (4.10)$$

and that

$$\begin{aligned} &\sum_{k=1}^d (\mu^k \bar{\varphi}_k d\hat{\varphi} - \mu^k \hat{\varphi}_k d\bar{\varphi}) \\ &= \sum_{k=1}^d \left[ d(\mu^k \bar{\varphi}_k \hat{\varphi}) - \mu^k \hat{\varphi} d\bar{\varphi}_k - \mu^k d\bar{\varphi}_k d\hat{\varphi} - (\mu^k \hat{\varphi} d\bar{\varphi})_k + \mu^k \hat{\varphi} d\bar{\varphi}_k + \mu_k^k \hat{\varphi} d\bar{\varphi} \right] \\ &= \sum_{k=1}^d \left[ d(\mu^k \bar{\varphi}_k \varphi) - \mu^k d\bar{\varphi}_k d\hat{\varphi} - (\mu^k \hat{\varphi} d\bar{\varphi})_k + \mu_k^k \hat{\varphi} d\bar{\varphi} \right]. \end{aligned} \quad (4.11)$$

Combining (4.10) and (4.11), we get the equality (4.9).  $\square$

Now we are in a position to prove Theorem 4.1.

*Proof of Theorem 4.1:* We first show that  $B$  is an admissible control operator with respect to the semigroup  $\{S(t)\}_{t \geq 0}$  generated by  $iA$ , i.e., we prove that there is a constant  $C > 0$  such that for any  $u \in L^2_{\mathbb{F}}(0, +\infty; U)$ ,

$$|\Psi_T u|_{L^2_{\mathcal{F}_T}(\Omega; H)} \leq C|u|_{L^2_{\mathbb{F}}(0, T; U)}$$

To proof the above inequality, we only need to establish the following inequality:

$$|\Psi_T^* \xi|_{L^2_{\mathbb{F}}(0, T; U)} \leq C|\xi|_{L^2_{\mathcal{F}_T}(\Omega; H)}, \quad \forall \xi \in L^2_{\mathcal{F}_T}(\Omega; H), \quad (4.12)$$

where  $C = C(T)$  is independent of  $\xi$ . To achieve this goal, we consider the following backward stochastic Schrödinger equation:

$$\begin{cases} dv + i\Delta v dt = -\bar{a}v dt - \bar{b}V dt + V dW(t) & \text{in } G \times (0, T), \\ v = 0 & \text{on } \Gamma \times (0, T), \\ v(T) = v_T & \text{in } G, \end{cases} \quad (4.13)$$

where  $v_T \in L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$ . By the classical theory of backward stochastic evolution equations (see [9]), we know that the equation (4.13) admits a unique solution  $(v, V) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G))) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G))$ .

Let  $w = A^{-1}v$ . Then  $w$  solves that

$$\begin{cases} dw + i\Delta w dt = -A^{-1}(\bar{a}v)dt - A^{-1}(\bar{b}V)dt + A^{-1}V dW(t) & \text{in } G \times (0, T), \\ w = 0 & \text{on } \Gamma \times (0, T), \\ w(T) = w_T = A^{-1}v_T & \text{in } G, \\ B^*v = B^*AA^{-1}v = -i\frac{\partial w}{\partial \nu} & \text{on } \Gamma_0 \times (0, T). \end{cases} \quad (4.14)$$

From the definition of  $A$ , we know that

$$\begin{aligned} w &\in C_{\mathbb{F}}([0, T]; L^2(\Omega; H_0^1(G))), \quad A^{-1}V \in L^2_{\mathbb{F}}(0, T; H_0^1(G)), \\ A^{-1}(\bar{a}v) &\in L^\infty_{\mathbb{F}}(0, T; H_0^1(G)), \quad A^{-1}(\bar{b}V) \in L^2_{\mathbb{F}}(0, T; H_0^1(G)). \end{aligned}$$

Further,

$$\begin{cases} |w|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; H_0^1(G)))} \leq C|v|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))}, \\ |A^{-1}V|_{L^2_{\mathbb{F}}(0, T; H_0^1(G))} \leq C|V|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))}, \\ |A^{-1}(\bar{a}v)|_{L^\infty_{\mathbb{F}}(0, T; H_0^1(G))} \leq C|a|_{L^\infty_{\mathbb{F}}(0, T; W_0^{1, \infty}(G))}|v|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))}, \\ |A^{-1}(\bar{b}V)|_{L^\infty_{\mathbb{F}}(0, T; H_0^1(G))} \leq C|b|_{L^\infty_{\mathbb{F}}(0, T; W_0^{1, \infty}(G))}|V|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))}. \end{cases} \quad (4.15)$$

Since  $\Gamma$  is  $C^2$ , there is a  $C^1$  vector field  $h = (h^1, \dots, h^d) : \overline{G} \rightarrow \mathbb{R}^d$  such that

$$h(x) = \nu(x) \quad \text{on } \Gamma, \quad |h(x)| \leq 1, \quad \forall x \in G.$$

Let us take  $\mu = h$  and  $\hat{\varphi} = w$  in (4.9). Integrating it in  $G \times (0, T)$  and taking the mathematical

expectation, we have that

$$\begin{aligned}
& -\mathbb{E} \int_0^T \int_G h \cdot \nabla \bar{w} [A^{-1}(\bar{a}v) + A^{-1}(\bar{b}V)] dxdt + \mathbb{E} \int_0^T \int_G h \cdot \nabla w [A^{-1}(a\bar{v}) + A^{-1}(b\bar{V})] dxdt \\
& = i\mathbb{E} \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \nu} \right|^2 dxdt + \int_G h \cdot \nabla \bar{w}(T)w(T)dx - \int_G h \cdot \nabla \bar{w}(0)w(0)dx \\
& \quad - i\mathbb{E} \int_0^T \int_G \sum_{j,k=1}^d (h_j^k + h_k^j) w_j \bar{w}_k dxdt + i\mathbb{E} \int_0^T \int_G (\nabla \cdot h) |\nabla w|^2 dxdt \\
& \quad + \mathbb{E} \int_0^T \int_G (\nabla \cdot h) w [i\Delta \bar{w} - A^{-1}(a\bar{v})dt - A^{-1}(b\bar{V})dt] dxdt - \mathbb{E} \int_0^T \int_G [h \cdot \nabla A^{-1}(\bar{V})] A^{-1}Z dxdt.
\end{aligned} \tag{4.16}$$

From (4.15), we know that

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \int_G h \cdot \nabla \bar{w} [A^{-1}(\bar{a}v) + A^{-1}(\bar{b}V)] dxdt \right| + \left| \mathbb{E} \int_0^T \int_G h \cdot \nabla w [A^{-1}(a\bar{v}) + A^{-1}(b\bar{V})] dxdt \right| \\
& \leq C(|w|_{L_{\mathbb{F}}^2(0,T;H_0^1(G))}^2 + |a|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}^2 |v|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2 + |b|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}^2 |V|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2) \\
& \leq C(|v|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))}^2 + |a|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}^2 |v|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2 \\
& \quad + |b|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}^2 |V|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2) \\
& \leq C|v_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
& \left| \int_G h \cdot \nabla \bar{w}(T)w(T)dx - \int_G h \cdot \nabla \bar{w}(0)w(0)dx \right| \\
& \leq C|w|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H_0^1(G)))}^2 \leq C|v|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))}^2 \leq C|v_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
& \left| -i\mathbb{E} \int_0^T \int_G \sum_{j,k=1}^d (h_j^k + h_k^j) w_j \bar{w}_k dxdt + i\mathbb{E} \int_0^T \int_G (\nabla \cdot h) |\nabla w|^2 dxdt \right| \\
& \leq C|w|_{L_{\mathbb{F}}^2(0,T;H_0^1(G))}^2 \leq C|w|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H_0^1(G)))}^2 \leq C|z|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))}^2 \leq C|z_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2,
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \int_G [h \cdot \nabla A^{-1}(\bar{V})] A^{-1}V dxdt \right| \\
& \leq C|A^{-1}V|_{L_{\mathbb{F}}^2(0,T;H_0^1(G))}^2 |V|_{L_{\mathbb{F}}^2(0,T;L^2(G))} \leq C|A^{-1}V|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2 \leq C|v_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2.
\end{aligned} \tag{4.20}$$

Further, since

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_G (\nabla \cdot h) w i \Delta \bar{w} dxdt \\
& = -\mathbb{E} \int_0^T \int_G \nabla \cdot (\nabla \cdot h) w i \nabla \bar{w} dxdt - \mathbb{E} \int_0^T \int_G (\nabla \cdot h) \nabla w i \nabla \bar{w} dxdt,
\end{aligned}$$

due to (4.15), we have that

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \int_G (\nabla \cdot h) w [i\Delta \bar{w} - A^{-1}(a\bar{v})dt - A^{-1}(b\bar{V})dt] dx dt \right| \\
& \leq C(|w|_{L_{\mathbb{F}}^2(0,T;H_0^1(G))}^2 + |a|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}^2 |v|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2 + |b|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}^2 |V|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))}^2) \\
& \leq C|v_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2.
\end{aligned} \tag{4.21}$$

From (4.16) to (4.21), we obtain that

$$\mathbb{E} \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \nu} \right|^2 dx dt \leq C|z_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2,$$

which implies that

$$\mathbb{E} \int_0^T \int_\Gamma |B^* z|^2 dx dt \leq C|z_T|_{L_{\mathcal{F}_T}^2(\Omega;H^{-1}(G))}^2. \tag{4.22}$$

Hence, we know that  $B$  is an admissible control operator.

Next, we prove that the input/output map is a bounded linear operator. Let  $\tilde{w} = A^{-1}y$ . Then  $\tilde{w}$  solves

$$\begin{cases} d\tilde{w} + i\Delta \tilde{w} dt = -i\Upsilon u dt + A^{-1}(ay)dt - A^{-1}(by)dW(t) & \text{in } G \times (0, +\infty), \\ \tilde{w} = 0 & \text{on } \Gamma \times (0, +\infty), \\ \tilde{w}(0) = \tilde{w}_0 = A^{-1}y_0 & \text{in } G, \\ \varphi = -i\frac{\partial \tilde{w}}{\partial \nu} & \text{on } \Gamma_0 \times (0, +\infty). \end{cases} \tag{4.23}$$

Since  $B$  is a admissible control operator, we know that  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$  and

$$|y|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H))} \leq C(|y_0|_{L_{\mathcal{F}_0}^2(\Omega;H)} + |u|_{L_{\mathbb{F}}^2(0,T;U)}). \tag{4.24}$$

From the definition of  $A$ , we know that

$$\tilde{w} \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H_0^1(G))), \quad A^{-1}(ay) \in L_{\mathbb{F}}^\infty(0, T; H_0^1(G)), \quad A^{-1}(by) \in L_{\mathbb{F}}^\infty(0, T; H_0^1(G)).$$

Further, thanks to (4.24), we have that

$$|\tilde{w}|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H_0^1(G)))} \leq C|y|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))} \leq C(|y_0|_{L_{\mathcal{F}_0}^2(\Omega;H)} + |u|_{L_{\mathbb{F}}^2(0,T;U)}),$$

$$\begin{aligned}
|A^{-1}(ay)|_{L_{\mathbb{F}}^\infty(0,T;H_0^1(G))} & \leq C|a|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}|y|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} \\
& \leq C(|y_0|_{L_{\mathcal{F}_0}^2(\Omega;H)} + |u|_{L_{\mathbb{F}}^2(0,T;U)})
\end{aligned}$$

and

$$\begin{aligned}
|A^{-1}(by)|_{L_{\mathbb{F}}^\infty(0,T;H_0^1(G))} & \leq C|b|_{L_{\mathbb{F}}^\infty(0,T;W_0^{1,\infty}(G))}|y|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} \\
& \leq C(|y_0|_{L_{\mathcal{F}_0}^2(\Omega;H)} + |u|_{L_{\mathbb{F}}^2(0,T;U)}).
\end{aligned}$$

According to these inequalities, similar to the proof of (4.22), we can obtain that

$$\mathbb{E} \int_0^T \int_\Gamma \left| \frac{\partial(-\Delta)^{-1}y}{\partial \nu} \right|^2 dx dt \leq \mathbb{E} \int_0^T \int_\Gamma \left| \frac{\partial \tilde{w}}{\partial \nu} \right|^2 dx dt \leq C(|y_0|_{L_{\mathcal{F}_0}^2(\Omega;H)}^2 + |u|_{L_{\mathbb{F}}^2(0,T;U)}^2). \tag{4.25}$$

This implies that the boundary observation operator in the system (4.1) is admissible and the system (4.1) is well-posed.  $\square$

## 5 Further comments and open problems

This paper is only a first and basic attempt to the study of well-posed linear stochastic system. In my opinion, there are many interesting and important problems in this topic. We present some of them here briefly:

- **Further properties for stochastic well-posed linear systems.**

Deterministic well-posed linear systems enjoy many deep and useful properties(see [24] for example). In this paper, we only investigate some very basic ones. It is an interesting and maybe challenging problem to study what kind of properties for deterministic well-posed linear systems also holds for stochastic well-posed linear systems.

- **The well-posedness of stochastic partial differential equations with boundary control/observation.**

The main motivation of introducing stochastic well-posed linear systems is to study the stochastic partial differential equations with boundary control/observation. In this paper, we only consider two special examples. It is well known that many deterministic partial differential equations with boundary control/observation are well-posed(see [5, 6, 7] and the rich references therein). It deserves to generalize there results for stochastic partial differential equations.

- **The study of the stochastic regular system.**

In the deterministic framework, there is another important concept, i.e., regular systems, in the study of infinite dimensional linear systems, which has very close relation to the well-posed system. One can also define the stochastic regular systems and study their properties. Some of them will appear in our forthcoming paper [18]. But there are still lots of problems should be studied.

- **The stabilization of stochastic control systems.**

Once we prove that a system is well-posed with  $U = \tilde{U}$ , then we can consider the stabilization of the system by the state feedback control. Such kind of problems are extensively studied for the deterministic control systems in the literature(see [12, 13, 19, 28] and the rich references therein). On the other hand, as far as we know, there is no result for the stochastic counterpart.

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